XXVIII Asian Pacific Mathematics Olympiad

March, 2016

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo.ommenlinea.org`*.*
Please do not disclose nor discuss the problems over online until that date. The use of calculators is not allowed.

Problem 1. We say that a triangle ABC is *great* if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC.

Prove that triangle ABC is great if and only if $\angle A = 90^\circ$ and $AB = AC$.

Problem 2. A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \ldots, a_{100}$ are non-negative integers that are not necessarily distinct.

Find the smallest positive integer n such that no multiple of n is a fancy number.

Problem 3. Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

Problem 4. The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer k such that no matter how Starways establishes its flights, the cities can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

Problem 5. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(z + 1)f(x + y) = f(xf(z) + y) + f(yf(z) + x),$$

for all positive real numbers x, y, z.