THE 1989 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let x_1, x_2, …, x_n be positive real numbers, and let

$$S = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$(1 + x_1)(1 + x_2)\cdots(1 + x_n) \le 1 + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \cdots + \frac{S^n}{n!}\ .$$

Question 2

Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

Question 3

Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1$, 2, and 3, suppose that B_n is the midpoint of A_nA_{n+1}, and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n, and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

Question 4

Let S be a set consisting of m pairs (a, b) of positive integers with the property that $1 \le a < b \le n$. Show that there are at least

$$4m \cdot \frac{(m - \frac{n^2}{4})}{3n}$$

triples (a, b, c) such that (a, b), (a, c), and (b, c) belong to S.

Question 5

Determine all functions f from the reals to the reals for which

(1) $f(x)$ is strictly increasing,
(2) $f(x) + g(x) = 2x$ for all real x,

where $g(x)$ is the composition inverse function to $f(x)$. (Note: f and g are said to be composition inverses if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x.)